FURTHER NOTE ON FACTORIZABLE CONTINUANTS.

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1. The main result contained in the previous note * was a generalisation embracing as special cases a theorem of Sylvester's and a theorem of Painvin's, Sylvester's theorem for the sixth order being

$$\begin{vmatrix} a & 1 & . & . & . & . \\ 5 & a & 2 & . & . & . \\ . & 4 & a & 3 & . & . \\ . & . & 3 & a & 4 & . \\ . & . & . & 2 & a & 5 \\ . & . & . & . & 1 & a \end{vmatrix} = (a^2 - 1^2)(a^2 - 3^2)(a^2 - 5^2).$$

The mode of treatment consisted in the removal of the factors, one by one, in the sequence

$$a + 5, \quad a + 3, \quad a + 1, \quad a - 1, \quad a - 3, \quad a - 5,$$

each removal being followed by a lowering of the order of the determinant.

2. To this an alternative course has since been found which, though not more effective in the matter of demonstration, has unexpectedly led to a new theorem of greater interest than that under consideration. Instead of removing one factor, advantage is taken of the fact that the continuant in question is centro-symmetric, and therefore is expressible as the product of two determinants of the third order, viz., in the form

$$\begin{vmatrix} a & 1 & . \\ 5 & a & 2 \\ . & 4 & a+3 \end{vmatrix} \cdot \begin{vmatrix} a & 1 & . \\ 4 & a & 2 \\ . & 5 & a-3 \end{vmatrix}.$$

* *Trans. S. A. Phil. Soc.*, xv. (1904), pp. 29–33.

The second of the two is not really different in type from the first, where the integers 1, 2, 3, 4, 5 in a manner *envelop* three equal elements; for if the first be $f(a)$, the second is

$$= - \begin{vmatrix} -a & -1 & . \\ -5 & -a & -2 \\ . & -4 & -a+3 \end{vmatrix} = - \begin{vmatrix} -a & 1 & . \\ 5 & -a & 2 \\ . & 4 & -a+3 \end{vmatrix}$$

$$= - f(-a).$$

Both, however, are essentially different from Sylvester's, and yet, from the nature of their connection with Sylvester's, are of necessity resolvable into linear factors. As a matter of fact the factors of the first are $a+5$, $a-3$, $a+1$: and there is suggested the general theorem

$$\begin{vmatrix} a & 1 & . & \cdots & . & . \\ 2n-1 & a & 2 & \cdots & . & . \\ . & 2n-2 & a & \cdots & . & . \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ . & . & . & \cdots & a & n-1 \\ . & . & . & \cdots & n+1 & a+n \end{vmatrix}_n = (a+\overline{2n-1}) \, . \, (a-\overline{2n-3}) \, . (a+\overline{2n-5}) \, . \, (a-\overline{2n-7}) \cdots \qquad (\alpha)$$

Trial being made upon this it is found that the method followed in the former paper suffices to effect the resolution, the set of line-multipliers employed being

$$\begin{array}{ccccccccc} 1, & 1, & 1, & 1, & 1, & 1, & 1, & 1, & \cdots \\ & 1, & -1, & 2, & -2, & 3, & -3, & 4, & \cdots \\ & & 1, & 1, & 3, & 3, & 6, & 6, & \cdots \\ & & & 1, & -1, & 4, & -4, & 10, & \cdots \\ & & & & 1, & 1, & 5, & 5, & \cdots \\ & & & & & 1, & -1, & 6, & \cdots \\ & & & & & & 1, & 1, & \cdots \\ & & & & & & & 1, & \cdots \\ & & & & & & & & \cdots \end{array} \qquad (S)$$

and every multiplier therefore of the form $C_{r+s,\, s+1}$. We are thus brought face to face with the problem of finding the most general continuant resolvable by means of this set of multipliers.

3. The first result obtained is: *If the continuant*

$$\begin{vmatrix} a & \beta_1 & . & \cdots \\ \gamma_1 & a+p & \beta_2 & \cdots \\ . & \gamma_2 & a+q & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

be resolvable into linear factors by means of the set of column-multipliers (S), *then*

$$\begin{aligned} \beta_1+\gamma_1 &= \beta_3+\gamma_3 = \beta_5+\gamma_5 = \ldots\ldots \\ \beta_2+\gamma_2 &= \beta_4+\gamma_4 = \beta_6+\gamma_6 = \ldots\ldots \end{aligned} \qquad \text{(I.)}$$

Of the truth of this there is no doubt, but a really short proof of it is much to be desired.

Calling the constant in the case of the odd suffixes s, and in the other case t, we may consequently write the continuant in the form

$$\begin{vmatrix} a & \beta_1 & . & \ldots\ldots \\ s-\beta_1 & a+p & \beta_2 & \ldots\ldots \\ . & t-\beta_2 & a+q & \ldots\ldots \\ \ldots & \ldots & \ldots & \ldots\ldots \end{vmatrix}_n ;$$

and when we perform upon this the first operation

$$\text{col}_1 + \text{col}_2 + \text{col}_3 + \ldots\ldots$$

we find, by reason of the removability of the factor $a+\beta_1$, that

$$\begin{aligned} p &= \beta_1 - s + \beta_1 - \beta_2, \\ q &= \beta_1 - t + \beta_2 - \beta_3, \\ &\ldots\ldots\ldots\ldots \end{aligned}$$

and that the resulting determinant can be lowered in order, being in fact

$$\begin{vmatrix} a+\beta_1-s-\beta_2 & \beta_2 & . & \ldots\ldots \\ t-\beta_2-\beta_1 & a+q & \beta_3 & \ldots\ldots \\ -\beta_1 & s-\beta_3 & a+r & \ldots\ldots \\ -\beta_1 & . & t-\beta_4 & \ldots\ldots \\ -\beta_1 & . & . & \ldots\ldots \\ \ldots & \ldots & \ldots & \ldots\ldots \end{vmatrix}_{n-1.}$$

By the performance on this of the second operation, viz.

$$\text{col}_1 - \text{col}_2 + 2\,\text{col}_3 - 2\,\text{col}_4 + 3\,\text{col}_5 - 3\,\text{col}_6 + \ldots\ldots$$

and the use of the fact that another factor is thus made removable we obtain a set of $n-2$ equations, which turn out to be sufficient for the complete solution of our problem. The equations involve all the β's and s and t, that is, in all $n+1$ quantities; and the key to the proper mode of treating the set lies in the selection of the three quantities in terms of which the others are most conveniently expressible. These three are s, t, β_1; but even with this fact in possession the solution merits some attention in detail, The equations are best written in the form

$$\left.\begin{array}{llllllll}
-\beta_1 - s + 2t & = 4\beta_2 - 3\beta_3 & & & & & & \\
\beta_1 + s & = 4\beta_2 + 3\beta_3 - 4\beta_4 & & & & & & \\
-\beta_1 - 2s + 4t & = 4\beta_2 & + 4\beta_4 - 5\beta_5 & & & & & \\
\beta_1 + 2s & = 6\beta_2 & & + 5\beta_5 - 6\beta_6 & & & & \\
-\beta_1 - 3s + 6t & = 6\beta_2 & & & + 6\beta_6 - 7\beta_7 & & & \\
\beta_1 + 3s & = 8\beta_2 & & & & + 7\beta_7 - 8\beta_8 & & \\
\multicolumn{8}{l}{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots}
\end{array}\right\}$$

and a little examination of them shows the need for considering two cases, viz., n even, and n odd.

4. Let us consider first the case where $n = 2m$. Here the last β is β_{2m-1}, and the last equation

$$\beta_1 + (m-1)s = 2m\beta_2 + (2m-1)\beta_{2m-1}.$$

Taking the first equation of the set, an equation derived by addition from the first two, an equation similarly derived from the first three, and so on, we obtain the following equivalent set

$$\left.\begin{array}{rl}
-\beta_1 - s + 2t = 4\beta_2 - 3\beta_3, & \\
& 2t = 8\beta_2 - 4\beta_4, \\
-\beta_1 - 2s + 6t = 12\beta_2 - 5\beta_5, & \\
& 6t = 18\beta_2 - 6\beta_6, \\
-\beta_1 - 3s + 12t = 24\beta_2 - 7\beta_7, & \\
& 12t = 32\beta_2 - 8\beta_8, \\
\dots\dots\dots\dots\dots\dots & \dots\dots\dots\dots \\
\multicolumn{2}{c}{2(1 + 2 + \dots + \overline{m-1})t = 2m^2\beta_2.}
\end{array}\right\}$$

From the last of the set we have

$$\beta_2 = \frac{m-1}{2m}t,$$

and substitution of this in the others gives

$$\begin{array}{ll}
\beta_3 = \frac{1}{3}\left(\beta_1 + s - \frac{1.2}{m}t\right), & \beta_4 = \frac{m-2}{2m}t, \\
\beta_5 = \frac{1}{5}\left(\beta_1 + 2s - \frac{2.3}{m}t\right), & \beta_6 = \frac{m-3}{2m}t, \\
\beta_7 = \frac{1}{7}\left(\beta_1 + 3s - \frac{3.4}{m}t\right), & \beta_8 = \frac{m-4}{2m}t, \\
\dots\dots\dots\dots\dots\dots & \dots\dots\dots \\
\beta_{2m-1} = \frac{1}{2m-1}(\beta_1 + \overline{m-1}.s - \overline{m-1}.t), & \beta_{2m-2} = \frac{1}{2m}t.
\end{array}$$

From these the γ's are readily calculable, and thereafter p, q, r, ... from the β's and γ's. Doing this, and putting, for neatness' sake, $a = \alpha + \frac{1}{2}t$ and $t = 2\tau$, we reach the following general theorem :—

The continuant of the 2*m*th *order*

$$\begin{vmatrix} A_1 & \beta_1 & \cdot & \cdots \\ \gamma_1 & A_2 & \beta_2 & \cdots \\ \cdot & \gamma_2 & A_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

is resolvable into linear factors if, putting M *for* $2b - s + \frac{1}{m}\tau$, *we have*

$$A_1 = \alpha + \tau,\quad A_{2\theta} = \alpha + \frac{\theta}{2\theta - 1}M,\quad A_{2\theta+1} = \alpha + \frac{\theta}{2\theta+1}M;$$

$$\beta_1 = b,\qquad \beta_{2\theta} = \frac{m-\theta}{m}\tau,\qquad \beta_{2\theta+1} = \frac{1}{2\theta+1}\left\{\theta s + b - \frac{2\theta(\theta+1)}{m}\tau\right\};$$

$$\gamma_1 = s - b,\quad \gamma_{2\theta} = \frac{m+\theta}{m}\tau,\qquad \gamma_{2\theta+1} = \frac{1}{2\theta+1}\left\{(\theta+1)s - b + \frac{2\theta(\theta+1)}{m}\right\};$$

the factors being

$$\begin{aligned} &\left(\alpha + b + \tau\right) \cdot \left(\alpha + b - s - \tau + \frac{2}{m}\tau\right) \\ &\cdot \left(\alpha + b + \tau - \frac{2}{m}\tau\right) \cdot \left(\alpha + b - s - \tau + \frac{4}{m}\tau\right) \\ &\cdot \left(\alpha + b + \tau - \frac{4}{m}\tau\right) \cdot \left(\alpha + b - s - \tau + \frac{6}{m}\tau\right) \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdot \left(\alpha + b + \tau - \frac{2m-2}{m}\tau\right) \cdot \left(\alpha + b - s - \tau + \frac{2m}{m}\tau\right). \end{aligned} \qquad \text{(II.)}$$

5. It is worth noting that

$$\begin{aligned} &\text{1st factor} = (1, 1) + (1, 2), \\ &\text{2nd factor} = (2, 2) - (1, 2) - (2, 3), \\ &\text{3rd factor} = (3, 3) + (2, 3) + (3, 4), \\ &\text{4th factor} = (4, 4) - (3, 4) - (4, 5), \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &(2m)\text{th factor} = (2m, 2m) - (2m-1, 2m), \end{aligned}$$

where the elements of the continuant are denoted by their place-names (1, 1), (1, 2), ... ; also, that the sum of any even-numbered factor and the factor immediately following it is $2a+2b-s$, thus giving for the sum of all the factors

$$(a+b+\tau) + (m-1)(2a+2b-s) + (a+b-s-\tau),$$

$$i.e., \quad m(2a+2b-s) + 2\tau.$$

Similarly we note that the sum of the first and second diagonal elements, and the sum of any other odd-numbered diagonal element and the element immediately following it, is $2a+\mathrm{M}$, the result being that the sum of all the diagonal elements is

$$m(2a+\mathrm{M})+\tau,$$

$$i.e., \quad m(2a+2b-s)+2\tau.$$

6. Putting s, which occurs only in the even-numbered factors of (II.), equal to $2b+\frac{1}{m}\tau$ we deduce from (II.) the identity

$$\left|\begin{array}{ccccccc} a+\tau & b & \cdot & \cdot & \cdot & \cdot & \ldots\ldots \\ b+\frac{1}{m}\tau & a & \frac{m-1}{m}\tau & \cdot & \cdot & \cdot & \ldots\ldots \\ \cdot & \frac{m+1}{m}\tau & a & b-\frac{1}{m}\tau & \cdot & \cdot & \ldots\ldots \\ \cdot & \cdot & b+\frac{2}{m}\tau & a & \frac{m-2}{m}\tau & \cdot & \ldots\ldots \\ \cdot & \cdot & \cdot & \frac{m+2}{m}\tau & a & b-\frac{2}{m}\tau & \ldots\ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{array}\right|_{2m}$$

$$\begin{aligned} = & \left(a+b+\tau\right) \cdot \left(a-b-\tau+\frac{1}{m}\tau\right) \\ & \cdot \left(a+b+\tau-\frac{2}{m}\tau\right) \cdot \left(a-b-\tau+\frac{3}{m}\tau\right) \\ & \cdot \left(a+b+\tau-\frac{4}{m}\tau\right) \cdot \left(a-b-\tau+\frac{5}{m}\tau\right) \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & \cdot \left(a+b+\tau-\frac{2m-2}{m}\tau\right) \cdot \left(a-b-\tau+\frac{2m-1}{m}\tau\right). \qquad \text{(III.)} \end{aligned}$$

Specialising still further by putting $b=2m-1$, and $\tau=2m$, we come back to the simple result (α) from which we started in § 2.

7. The similar investigation of the case where the continuant is of odd order leads to the following theorem :—

The continuant of the $(2m-1)$th *order*

$$\begin{vmatrix} A_1 & \beta_1 & \cdot & \cdots\cdots \\ \gamma_1 & A_2 & \beta_2 & \cdots\cdots \\ \cdot & \gamma_2 & A_3 & \cdots\cdots \\ \cdots & \cdots & \cdots & \cdots\cdots \end{vmatrix}$$

is resolvable into linear factors, if, putting

$$R \equiv s + \frac{1}{m-1}b, \quad N \equiv \frac{2m-1}{m}\left(\frac{2m}{m-1}b - R\right)$$

we have

$$A_1 = a + \tau, \quad A_{2\theta} = a + \frac{\theta}{2(2\theta-1)}N, \quad A_{2\theta+1} = a + \frac{\theta}{2(2\theta+1)}N;$$

$$\beta_1 = b, \qquad \beta_{2\theta} = \tau - \frac{\theta}{2m}R, \qquad \beta_{2\theta+1} = \frac{m-\theta-1}{m(2\theta+1)}\left\{\frac{m}{m-1}b + \theta R\right\};$$

$$\gamma_1 = s - b, \quad \gamma_{2\theta} = \tau + \frac{\theta}{2m}R, \qquad \gamma_{2\theta+1} = \frac{m+\theta}{m(2\theta+1)}\left\{(\theta+1)R - \frac{m}{m-1}b\right\};$$

the factors being

$$\begin{aligned}
&\left(a+b+\tau\right) \cdot \left(a+b-\tau-s+\frac{1}{m}R\right) \\
\cdot\, &\left(a+b+\tau-\frac{1}{m}R\right) \cdot \left(a+b-\tau-s+\frac{2}{m}R\right) \\
\cdot\, &\left(a+b+\tau-\frac{2}{m}R\right) \cdot \left(a+b-\tau-s+\frac{3}{m}R\right) \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\cdot\, &\cdots\cdots\cdots\cdots\cdots \cdot \left(a+b-\tau-s+\frac{m-1}{m}R\right) \\
&\left(a+b+\tau-\frac{m-1}{m}R\right).
\end{aligned} \qquad \text{(IV.)}$$

Here we have to note that

$$A_{2\theta-1} + A_{2\theta} = 2a + \tfrac{1}{2}N,$$

and hence that

$$A_1 + A_2 + \ldots + A_{2m-1} = (2m-1)(a+b) - (m-1)s + \tau,$$

—a result readily verified by looking at the sum of the factors.

There are at least two interesting special cases, viz., when $s = -\frac{1}{m-1}b$, and when $s = -\frac{2m-1}{m-1}b$. With the former substitution we reach the identity

$$\left|\begin{array}{ccccccc} a+\tau & (m-1)\beta & . & . & . & . & \cdots\cdots \\ -m\beta & a+\frac{1}{2}\mathrm{H} & \tau & . & . & . & \cdots\cdots \\ . & \tau & a+\frac{1}{6}\mathrm{H} & \frac{1}{3}(m-2)\beta & . & . & \cdots\cdots \\ . & . & -\frac{1}{3}(m+1)\beta & a+\frac{1}{3}\mathrm{H} & \tau & . & \cdots\cdots \\ . & . & . & \tau & a+\frac{1}{5}\mathrm{H} & \frac{1}{5}(\mathrm{M}-3)\beta & \cdots\cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}\right|_{2m-1}$$

$$= (a+\tau+\overline{m-1}.\beta)^m(a-\tau+m\beta)^{m-1}, \text{ where } \mathrm{H} = (4m-2)\beta; \qquad \text{(V.)}$$

and with the latter the identity

$$\left|\begin{array}{ccccccc} a+\tau & (m-1)\beta & . & . & . & . & \cdots\cdots \\ m\beta & a & \tau-\beta & . & . & . & \cdots\cdots \\ . & \tau+\beta & a & (m-2)\beta & . & . & \cdots\cdots \\ . & . & (m+1)\beta & a & \tau-2\beta & . & \cdots\cdots \\ . & . & . & \tau+2\beta & a & (m-3)\beta & \cdots\cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}\right|_{2m-1}$$

$$= (a+\tau+\overline{m-1}.\beta)(a-\tau-\overline{m-2}.\beta)(a+\tau+\overline{m-3}.\beta)\cdots\cdots \qquad \text{(VI.)}$$

which degenerates into the original on putting $\beta = 2\mathrm{B}$, and $\tau = (2m-1)\mathrm{B}$.

8. On returning to the two main results, (II.) and (IV.), it is readily seen that, although the two continuants seem to be functions of four variables, viz., a, b, s, τ, this is not really so, because the right-hand members are expressible in terms of three variables only. In the case of the even-ordered continuant these latter variables are $a+b$, s, τ, and in the other case $a+b+\tau$, $s+\frac{1}{m-1}b$, $s+2\tau$. Putting therefore $\tau=0$, $s=0$, $a=-b$ in (II.), and $s=-2\tau$, $b=2(m-1)\tau$, $a=-(2m-1)\tau$ in (IV.), we obtain the corresponding nil-factor continuants, that is to say, continuants whose matrices may be added to the matrices of the continuants in (II.) and (IV.) without affecting the identities. For example, knowing that the continuant

$$\left|\begin{array}{cccc} a+4 & 3 & . & . \\ 5 & a & 2 & . \\ . & 6 & a & 1 \\ . & . & 7 & a \end{array}\right| = (a+7)(a-5)(a+3)(a-1),$$

we can assert that, ξ being any quantity whatever, the derived continuant

$$\begin{vmatrix} a+4+\xi & 3-\xi & . & . \\ 5+\xi & a-\xi & 2 & . \\ . & 6 & a+\frac{1}{3}\xi & 1-\frac{1}{3}\xi \\ . & . & 7+\frac{1}{3}\xi & a-\frac{1}{3}\xi \end{vmatrix}$$

has exactly the same value. Thus, taking for ξ the value 3, which causes two of the elements to vanish, we have the new continuant

$$= (a+7)(a-1) \,.\, \begin{vmatrix} a-3 & 2 \\ 6 & a+1 \end{vmatrix},$$

$$= (a+7)(a-1) \,.\, (a-5)(a+3).$$

The elements of the nil-factor continuants corresponding to the set of column-multipliers (S) *of* § 2 *are for the* $(2m)$*th order*

$$\left.\begin{array}{lll} A_1 = \xi, & A_{2\theta} = -\dfrac{1}{2\theta-1}\xi, & A_{2\theta+1} = \dfrac{1}{2\theta+1}\xi, \\ \beta_1 = -\xi, & \beta_{2\theta} = 0, & \beta_{2\theta+1} = -\dfrac{1}{2\theta+1}\xi, \\ \gamma_1 = \xi, & \gamma_{2\theta} = 0, & \gamma_{2\theta+1} = \dfrac{1}{2\theta+1}\xi, \end{array}\right\} \quad \text{(VII.)}$$

and for the $(2m-1)$*th order*

$$\left.\begin{array}{lll} A_1 = -2(m-1)\xi, & A_{2\theta} = \dfrac{2m-1}{2\theta+1}\xi, & A_{2\theta+1} = -\dfrac{2m-1}{2\theta+1}\xi, \\ \beta_1 = 2(m-1)\xi, & \beta_{2\theta} = \xi, & \beta_{2\theta+1} = \dfrac{2(m-1-\theta)}{2\theta+1}\xi, \\ \gamma_1 = -2m\xi, & \gamma_{2\theta} = \xi, & \gamma_{2\theta+1} = -\dfrac{2(m+\theta)}{2\theta+1}\xi, \end{array}\right\} \text{(VIII.)}$$

9. When the nil-factor matrix (VII.) is added to the matrix of (II.) in §4, the element in the place (1, 2) is $b-\xi$, and the element in the place $(2m-1, 2m)$ is $\dfrac{1}{2m-1}\left\{b+\overline{m-1}\,.\,s-\overline{m-1}\,.\,\tau\right\} - \dfrac{1}{2m-1}\xi$. Both elements will therefore vanish when $x = b$ and $s = \tau$, and it will be possible to remove the first and last factors from both sides, thus giving a new continuant of the $(2m-2)$th order equal to the

product of the remaining factors. Doing this, and writing A for $a+b-\tau$, we arrive at the identity

$$\left|\begin{array}{cccccc} A+\frac{1-m}{m}\tau & \frac{m-1}{m}\tau & . & . & . & \cdots\cdots \\ \frac{m+1}{m}\tau & A+\frac{1+m}{3m}\tau & \frac{2}{3}\cdot\frac{m-2}{m}\tau & . & . & \cdots\cdots \\ . & \frac{4}{3}\cdot\frac{m+1}{m}\tau & A+\frac{2-m}{3m}\tau & \frac{m-2}{m}\tau & . & \cdots\cdots \\ . & . & \frac{m+2}{m}\tau & A+\frac{2+m}{5m}\tau & \frac{4}{5}\cdot\frac{m-3}{m}\tau & \cdots\cdots \\ . & . & . & \frac{6}{5}\cdot\frac{m+2}{m}\tau & A+\frac{3-m}{5m}\tau & \cdots\cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}\right|_{2m-2}$$

$$\begin{aligned} &=\left(A-2\tau+\frac{2}{m}\tau\right)\cdot\left(A+2\tau-\frac{2}{m}\tau\right)=\left\{A^2-\left(\frac{m-1}{m}.2\tau\right)^2\right\}\cdots\left\{A^2-\left(\frac{1}{m}.2\tau\right)^2\right\} \\ &\quad\cdot\left(A-2\tau+\frac{4}{m}\tau\right)\cdot\left(A+2\tau-\frac{4}{m}\tau\right) \\ &\quad\cdots\cdots\cdots\cdots\cdots\cdots \qquad\qquad\qquad\qquad \text{(IX.)} \\ &\quad\cdot\left(A-2\tau+\frac{2m-2}{m}\tau\right)\cdot\left(A+2\tau-\frac{2m-2}{m}\tau\right) \end{aligned}$$

Here the general expressions for the elements are

$$\left.\begin{aligned} A_{2\theta-1} &= A+\frac{\theta-m}{(2\theta-1)m}\tau, & A_{2\theta} &= A+\frac{\theta+m}{(2\theta+1)m}\tau, \\ \beta_{2\theta-1} &= \frac{m-\theta}{m}\tau, & \beta_{2\theta} &= \frac{2\theta}{2\theta+1}\cdot\frac{m-\theta-1}{m}\tau, \\ \gamma_{2\theta-1} &= \frac{m+\theta}{m}\tau, & \gamma_{2\theta} &= \frac{2(\theta+1)}{2\theta+1}\cdot\frac{m+\theta}{m}\tau; \end{aligned}\right\}$$

but as the set of column-multipliers necessary for effecting the resolution is easily ascertained to be

$$\begin{array}{cccccccc} 1, & -1, & 2, & -2, & 3, & -3, & \cdots\cdots \\ & 1, & 1, & 3, & 3, & 6, & \cdots\cdots \\ & & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

that is to say, the set (S) deprived of its first line, a more general theorem than (IX.) is obtainable by means of the procedure of §§ 3, 4.

When τ is put equal to m in (IX.) we obtain a result which on one side closely resembles Sylvester's.

10. On glancing back at the preceding paragraphs it will be seen that the new results obtained have arisen naturally out of an investigation of the accidentally discovered identity (a) of § 2, where the continuant is of such a form that it may be not inaptly described as consisting of a diagonal of identical elements ensheathed by the first $2n-1$ integers, and where the factors into which it is resolvable form not one equidifferent series but two.

Now there is another factorizable continuant, having some of these characteristics, which is equally interesting in itself and equally full of promise as a base for investigation, viz.,

$$\left|\begin{array}{cccccc} a & \frac{(2n-1)\beta}{2n-1} & \cdot & \cdots\cdots & \cdot & \cdot \\ \frac{1.(\beta+2n-2)}{3-2n} & a & \frac{(2n-2)(\beta+1)}{2n-3} & \cdots\cdots & \cdot & \cdot \\ \cdot & \frac{2(\beta+2n-3)}{5-2n} & a & \cdots\cdots & \cdot & \cdot \\ \cdot & \cdot & \frac{3(\beta+2n-4)}{7-2n} & \cdots\cdots & \cdot & \cdot \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdot & \cdot & \cdot & \cdots\cdots & a & \frac{(n+1)(\beta+n-2)}{3} \\ \cdot & \cdot & \cdot & \cdots\cdots & \frac{(n-1)(\beta+n)}{-1} & a+\frac{n(\beta+n-1)}{1} \end{array}\right|$$

$$= (a+\beta)\,(a+\beta+2)\,(a+\beta+4)\,\ldots\ldots\,(a+\beta+2n-2). \qquad \text{(X.)}$$

Here in the sheath enclosing the diagonal of a's we have, as in the previous case, the integers 1, 2, 3, ..., $2n-1$ going round the one way, but we have also accompanying them as multipliers the quantities β, $\beta+1$, $\beta+2$, ..., $\beta+2n-2$ going round the reverse way, and accompanying the latter as divisors the integers $2n-1$, $2n-3$, ..., 1, -1, -3, ..., $-(2n-3)$.

Using the fact that the right-hand side of (X.) is a function not of a and β separately, but of $a+\beta$, we find the corresponding nil-factor continuant to be

$$\left|\begin{array}{cccccc} -\xi & \frac{2n-1}{2n-1}\xi & \cdot & \cdots\cdots & \cdot & \cdot \\ \frac{1}{3-2n}\xi & -\xi & \frac{2n-2}{2n-3}\xi & \cdots\cdots & \cdot & \cdot \\ \cdot & \frac{2}{5-2n}\xi & -\xi & \cdots\cdots & \cdot & \cdot \\ \cdot & \cdot & \frac{3}{7-2n}\xi & \cdots\cdots & \cdot & \cdot \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdot & \cdot & \cdot & \cdots\cdots & -\xi & \frac{n+1}{3}\xi \\ \cdot & \cdot & \cdot & \cdots\cdots & \frac{n-1}{-1}\xi & -\xi+\frac{n}{1}\xi \end{array}\right|, \qquad \text{(XI.)}$$

and we are thus able to simplify the original identity into

$$\begin{vmatrix} A & \frac{1.2n}{2n-1} & . & \cdots\cdots & . & . \\ \frac{(2n-1).2}{3-2n} & A & \frac{2(2n-1)}{2n-3} & \cdots\cdots & . & . \\ . & \frac{(2n-2).3}{5-2n} & A & \cdots\cdots & . & . \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ . & . & . & \cdots\cdots & A & \frac{(n-1)(n+2)}{3} \\ . & . & . & \cdots\cdots & \frac{(n+1)n}{-1} & A+\frac{n(n+1)}{1} \end{vmatrix}$$

$$= (A+2)(A+4)(A+6) \ldots\ldots (A+2n). \qquad \text{(XII.)}$$

Unfortunately the set of column-multipliers necessary for effecting this resolution is not simple, the first line of it being

$$1, \quad 2-\frac{1}{n}, \quad 3-\frac{3}{n}, \quad 4-\frac{6}{n}, \quad \ldots\ldots, \quad n-\frac{n(n-1)}{2n}.$$

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